

# Phys. 235 Lecture Notes - Week 1

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## Purpose

Study the transport properties in stochastic magnetic systems where magnetic field lines deviate from their unperturbed stationary orbits. For this particular example, Hamiltonian dynamics as well as statistical mechanics foundations are used to investigate the trajectories and paths of '*braided*' perturbed magnetic field lines in a plasma, that are chaotic and random.

## 1 Introduction and Basic classical mechanics concepts

In order to describe the dynamics of any Hamiltonian system, one can use the Hamiltonian function  $H(p, q)$ , where  $p$  and  $q$  are the system momentum and position in the phase space respectively [4]. When this Hamiltonian is time independent ( $dH/dt = 0$ ), the energy of the system is conserved and  $E(p, q)$  is said to be a constant of motion. Note that:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

and

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

When using the appropriate generating function  $S(\bar{p}, q, t)$ , a canonical change of variables from  $(p, q) \rightarrow (\bar{p}, \bar{q})$  can be performed, while preserving the Hamiltonian form. In terms of  $S(\bar{p}, q, t)$ , the change of variables is specified as:

$$\bar{q} = \frac{\partial S}{\partial \bar{p}}$$
$$p = \frac{\partial S}{\partial q}$$

and  $p$  and  $q$  are said to be canonically conjugate. This change of variables preserves the Hamiltonian form and is performed to make the study of the system dynamics easier.

If we have an integrable system, that is a system with a time independent Hamiltonian ( $dH/dt = 0$ ) and global constants of motion that are independent of each other, we can perform a canonical change of variables from  $(p, q) \rightarrow (J, \theta)$  where  $\theta$  is the angle variable and  $J$  is the action variable defined as  $J = \oint p \cdot dq$ . Note that both  $j$  and  $\theta$  are multicomponent variables so it makes sense to write the equation in vector notation. This change of variables gives:

$$\frac{d\vec{\theta}}{dt} = \frac{\partial H}{\partial \vec{J}} = \vec{w}(\vec{J})$$

where  $\vec{w}(\vec{J})$  is the angular velocity. The new Hamiltonian is now only  $\vec{J}$  dependent,  $H = H(\vec{J})$ , and the action variable is conserved along certain unperturbed orbits:  $d\vec{J}/dt = \partial H / \partial \vec{\theta} = 0$ .

In a 2-torus periodic cylinder system such as a tokamak, orbits with rational angular velocity ratio  $w_1/w_2 = p/q$  are closed unperturbed surfaces, along which particle resonance can occur. However, if this ratio  $p/q$  is irrational, trajectories are open and particles follow ergodic paths that tend to fill the whole torus space.

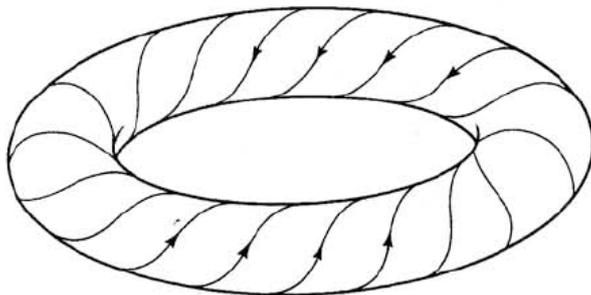


Figure 1: Rational surface in a 2-torus tokamak.

In the more general perspective, a canonical change of variable comes in handy when investigating the perturbation effects on the system dynamics using the KAM theorem. When we perturb the Hamiltonian  $H_0(\vec{J})$  by adding a small quantity  $\epsilon H_1(\vec{J}, \vec{\theta})$  with  $\epsilon$  small, we should expect the particles orbits defined by the appropriate constants of motion and the corresponding closed rational surfaces  $w_1/w_2 = p/q$  to be destroyed as soon as  $\epsilon \neq 0$ . Our purpose is to try to recover the new closed orbits set that corresponds to the perturbed Hamiltonian  $H(\vec{J}, \vec{\theta}) = H_0(\vec{J}) + \epsilon H_1(\vec{J}, \vec{\theta})$  using the KAM theorem results. If these closed new orbits exist, there must be a way to do a canonical change of variables so to express this same Hamiltonian  $H(\vec{J}, \vec{\theta})$  in terms of some new constant of motion  $\vec{J}'$ ; we are basically saying that  $H(\vec{J}, \vec{\theta}) = H(\vec{J}')$  for a canonical change of variable  $(\vec{J}, \vec{\theta}) \rightarrow (\vec{J}', \vec{\theta}')$ . The generating function  $S$  allowing such a change of variables is expressed in terms of  $\vec{J}'$  and  $\vec{\theta}$ . We furthermore expand the generating function  $S = S_0 + \epsilon S_1$

and the Hamiltonian  $H_1$  expressions as Fourier series and write them as:

$$S_1 = \sum_m S_{1,m}(\vec{J}') \exp(i\vec{m} \cdot \vec{\theta})$$

$$H_1 = \sum_m H_{1,m}(\vec{J}') \exp(i\vec{m} \cdot \vec{\theta})$$

where  $\vec{m}$  is a vector of integers, we find that the new perturbed surfaces approximate the unperturbed orbits of the integrable system. For such type of orbit self preservation, the change in the generating function  $\Delta S = S_1$  needs to be equal to:

$$\Delta S = \frac{\epsilon H_1(\vec{J}')}{\vec{m} \cdot \vec{\omega}}$$

$\Delta S$  is clearly not defined for  $\vec{m} \cdot \vec{\omega} = 0$ . Such a scenario referred to as the small denominator problem is responsible of creating orbits resonance islands where trajectories braid from one X point to the other. Around these islands, orbits become less and less perturbed as they get further away from the O point. If two or more islands are present close to each other, chaos fills in the separating volume by non-ergodic turbulent mixing. This happens when

$$\frac{\Delta \vec{J}_\alpha + \Delta \vec{J}_\beta}{\| \vec{J}_{0,\alpha} - \vec{J}_{0,\beta} \|} > 1$$

One can use Quasi linear or the Fokker-Planck theory to describe the particles behavior in those regions, shown to be of diffusive nature [6, 7].

## 2 Magnetic field lines mixing

A prime example of the perturbed orbits and stochastic transport problem is that of the magnetic field lines perturbation in tokamaks. Ideally, unperturbed  $\vec{B}$  lines lie on nested sets of magnetic flux surfaces and act to keep charged particles confined in certain regions inside the torus. Unfortunately, unavoidable symmetry breaking leads to magnetic field line perturbation. Magnetic field irregularities result in significant changes in the magnetic topology, lead to the birth of magnetic islands or even regions with destroyed magnetic surfaces, particularly near the separatrix, and one finds himself facing the perturbed/unperturbed orbits problem mentioned above. Electron thermal conductivity appearing to be sensitive to this effect, it would be insightful to look at the change of transport proprieties in a stochastic magnetic field line configuration.

In Ref. [7], the authors used Vlasov equation to study deviations of field lines from their unperturbed resonant surfaces and to investigate the system response to the adjacent resonant perturbations overlapping on top of each other. The authors tried to qualitatively describe the type of random motion

according to which the chaotic magnetic field takes over and fills in the space. The perturbed magnetic field is:

$$\vec{B} = \tilde{B}_r \hat{r} + (B_\theta(r) + \tilde{B}_\theta) \hat{\theta} + B_0 \hat{z} \quad (1)$$

The equation of magnetic fields ( $\nabla \cdot \vec{B} = 0$ ) generates then:

$$\frac{dr}{\tilde{B}_r} = \frac{dz}{B_z} = \frac{rd\theta}{B_\theta(r) + \tilde{B}_\theta} \quad (2)$$

and the Liouville equation  $\vec{B} \cdot \nabla f = 0$  gives:

$$\frac{\partial f}{\partial \theta} + \frac{rB_r}{B_\theta} \cdot \frac{\partial f}{\partial r} + \frac{rB_z}{B_\theta} \cdot \frac{\partial f}{\partial z} = 0 \quad (3)$$

where  $f$  is the probability distribution function of orbits in  $(r, \theta, z)$ .

#### Side Note: Kubo number and some of its proprieties

Before going deeper in the discussion, it is worth mentioning that cases where the Kubo number (defined as the ratio of radial excursion from the unperturbed case to the radial correlation length:  $Ku = \delta r / \Delta r$ ) is small are of particular interest. This low  $Ku$  number limit forces us to consider only small perturbations for the time being. From Eq.(2), the transverse excursion is  $dr = dz \cdot (\tilde{B}_r / B_0)$  and:

$$\delta r = l_{ac} \cdot (\tilde{B}_r / B_0) \quad (4)$$

where  $l_{ac} = \Delta |k_{\parallel}|^{-1}$  is the inverse spatial bandwidth of the parallel perturbation spectrum. Alternatively, if we consider the ratio of linear to nonlinear terms in Eq.(2), this ratio boils down to:

$$\frac{l_{ac}}{\Delta r} \cdot \frac{\tilde{B}_r}{B_0}$$

which in fact is the Kubo number.

- **$Ku < 1$ : It is a chaotic quasi linear diffusive transport case. The weakly scattered particles experience multiple kicks in a single radial scattering step size  $\delta r$ . In this case, particles easily lose their memory before completing the circumnavigation in a trough of one large scale perturbation wave, which makes the linear theory applicable.**
- **$Ku > 1$ : This is transport case where linear theory fails because of the strong scattering of the particles. In one kick, particles cover a distance greater than  $\delta r$ . The fact that particles are trapped plays a major role in determining the nonlinear effects leading to percolation.**

- **For  $Ku \sim 1$ : This is the case that defines the mixing length in fusion systems. It translates a balance between the eddy turnover time in a system and the turbulence auto correlation time. Another way of interpreting a  $Ku = 1$  value would be to say that the particle lifetime limited by the nonlinear effects of the perturbation is equal to the time particles take to bounce back while traveling in the trough of a wavelength perturbation. In this case, turbulent states are qualified as stationary.**

Going back to Eq.(4) in our problem, the authors in ref. [7] used Quasi linear theory to show that in the case of perturbed magnetic field lines, the random motion of these lines is diffusive. When resonance islands appear, perturbations around them spread out. Of course those perturbations start fading when they get further away from the islands. However, when adjacent resonance surfaces overlap, magnetic flux lines diffuse from one resonant structure to the other according to a Brownian motion which results in surface flux distortion and chaos irreversible non ergodic mixing. The authors used Liouville equation and quasi-linear theory to write:

$$\frac{\partial \langle f \rangle}{\partial z} = \frac{\partial}{\partial r} [D_M \frac{\partial \langle f \rangle}{\partial r}] \quad (5)$$

Here the radial magnetic field  $\tilde{B}_r$  and the magnetic diffusion coefficient  $D_M$  are:

$$\tilde{B}_r = \sum_{m,n} B_{m,n}(r) e^{i(m\theta - n\phi)} = \sum_{k_z, k_\theta} \tilde{B} e^{i(k_\theta y - k_z z)} \quad (6)$$

$$D_M = \frac{1}{4} \cdot l_{ac} \cdot \langle \frac{\tilde{B}_r}{B_\theta} \rangle = \frac{1}{4} \cdot l_{ac} \cdot \langle (\frac{\delta B}{B_0})^2 \rangle \quad (7)$$

The average is an ensemble average and Eq.(7) is taken from ref.[8]. Eq.5 has the form of a diffusion equation and yields a mean radial diffusive flux  $\Gamma(r)$  due to stochastic wandering of the lines.

$$\Gamma(r) = -D_M \frac{\partial \langle f \rangle}{\partial r}$$

In the  $k$ -space, the expression of this flux is:

$$\Gamma = -\tilde{B}_{r(\parallel)} \frac{\partial \langle f \rangle}{\partial r} = \sum_k \left| \frac{\delta \tilde{B}_r(k)}{B_0} \right|^2 \pi \delta(k_\parallel) = \langle (\frac{\delta B_r}{B_0})^2 \rangle l_{ac}$$

where  $\sum_k = \sum_{m,n}$  in the expression of  $\tilde{B}_r$ . This means that a diffusion of  $B$  lines along a distance  $z$  imposes a  $\delta r$  value written as:

$$\langle (\delta r)^2 \rangle = D_M \cdot z \sim l_{ac} \cdot (\frac{\tilde{B}_r}{B})^2 \cdot z \quad (8)$$

( $\delta r$  is the radial excursion from the unperturbed orbits or lines).

Looking back at equations (7) and (8), in order to find the diffusion coefficient  $D_M$ , expressions of  $\tilde{B}_r$  and  $l_{ac}$  are needed (See Eq.7). This requires an expression of  $\delta r$  (See Eq.8).

$$\delta r = \int_0^l (\tilde{B}_r/B_0) dz$$

This in turn begs the question: what is  $l$  really equal to? In our system we have three scales of length;

- $l_{ac}$  = the coherence or the memory line length for scattering fields, related to the auto correlation time  $\tau_{ac}$  in 1D.
- $l_{mfp}$  = mean free path according to which one can have either a collisional ( $l_{ac} < l_{mfp} < l_c$ ) or a collisionless regime ( $l_{ac} < l_{mfp} < l_c$ ).
- $l_c$  = magnetic field lines decorrelation length which is analogous to  $l_{ac}$  and emerges from considering the decorrelation of trajectories from the linear unperturbed ones due to field scattering.

While the first two are well defined, further investigation of  $l_c$  is needed and was indeed first initiated by G. I. Taylor, Kelvin and Dupree.

In cylindrical geometry, considering a deviation ( $\delta r, \delta\theta$ ) from the unperturbed orbits occurs, and using the relation  $r.(d\theta/dz) = (\tilde{B}_r/B_0)$ , we have:

$$r.\frac{d\delta\theta}{dz} = \delta r.\frac{B'_\theta}{B_0}$$

where the derivative results from Taylor expanding  $B_\theta$  at a value  $\delta r$ . Setting  $\delta y = r\delta\theta$ :

$$\langle (\delta y)^2 \rangle = \langle \int_0^z (\frac{B'_\theta}{B_0}.\delta r.dz')^2 \rangle = (\frac{B'_\theta}{B_0}.z)^2. \langle (\delta r)^2 \rangle = (\frac{B'_\theta}{B_0})^2.D_M.z^3$$

because  $\langle (\delta r)^2 \rangle = D_M.z$ . Now since  $\overline{k_\theta^2}.\langle (\delta y)^2.l_c \rangle \sim 1$ , we have:

$$l_c \sim (\overline{k_\theta^2}.\frac{B'_\theta}{B_0})^2.D_M)^{-1/3} = (\frac{\overline{k_\theta^2}.D_M}{L_S^2})^{-1/3} \quad (9)$$

where  $L_S$  is a gradient scale length of the magnetic field. When the ratio  $l_{ac}/l_c$  is less than one,  $Ku < 1$  and the quasi linear approximation is valid. Alternatively, in the opposite case where  $Ku > 1$ , one can still use the  $\nabla.\vec{B} = 0$  equation, which allows to write  $\vec{B}$  as a function  $\Psi$ :

$$\frac{\partial x}{\partial z} = \frac{\partial \Psi}{\partial y}, \frac{\partial y}{\partial z} = -\frac{\partial \Psi}{\partial x}$$

When  $z$  is replaced by the time  $t$ , the previous two equations can be thought of as equations of motion. But regardless of this analogy, field lines go along constant  $\Psi$  lines. The behavior of the  $\Psi = \text{const}$  lines for a random  $\Psi$  function has been analyzed in the problem of current percolation in random inhomogeneous solids [3].

### 3 Electron thermal conductivity

What we want is to calculate the electron thermal conductivity in a chaotic stochastic magnetic field lines configuration. In ref.[6], electrons motion and spreading away from their unperturbed orbits is shown to be happening according to a Brownian motion with a corresponding thermal diffusivity equal to:

$$\chi_r = \frac{\langle (\delta r)^2 \rangle}{2t}$$

where  $\langle (\delta r)^2 \rangle = 2.L.D_M = 2D_M(\chi_{\parallel}t)^{1/2}$  is the average squared radial displacement traveled by the electron during a time  $t$  and described by the diffusion equation. Here  $L$  is the distance in the  $z$ -direction. Clearly, to get  $\chi$  expression, we need to find  $\langle (\delta r)^2 \rangle$ . But before trying to determine the values of  $\langle (\delta r)^2 \rangle$  and  $\chi$ , one needs to distinguish between two cases or regimes. This distinction is dictated by the corresponding length orderings:

Case 1:  $l_{ac} < l_c < l_{mfp}$ : the so called collisionless case

When calculating the particle heat diffusion coefficient, one needs to consider diffusion in both parallel and perpendicular directions. Simply put, without any perpendicular diffusion, particles will hit each other and then go back to their initial position according to the same unperturbed orbit ( $l_c < l_{mfp}$  means that the orbit remains unchanged and unperturbed within one collision time) (Left picture in fig.2).

If this is correct (that is the absence of any sort of perpendicular scattering), particles would remain on the same orbit and experience only parallel collisions that kick them back and forth. Clearly this is not the case because such a scenario would mean a complete lack of radial transport. With necessary perpendicular particle diffusion in mind, one can adopt some sort of a perpendicular resolution, which although small, but not be equal to zero (like a Finite Larmor Radius or electron gyro-radius for example), to find the appropriate perpendicular diffusion coefficient that allows for the particles to wander away and to be kicked off the unperturbed magnetic fields securing this way a radial particle transport. This perpendicular coarse graining is the source of irreversibility in the diffusion process.

As an example, let us consider a disk of radius  $\rho_e$  moving in the torus and let us study its dynamics within a time  $t < \tau_{collision}$ . Because of the phase

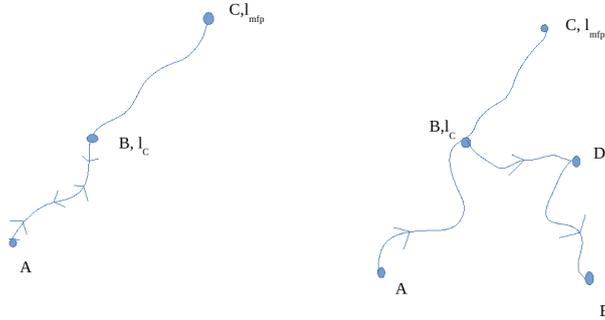


Figure 2: Particle trajectories without and with perpendicular diffusion. Arrows represent particle directions of motion.

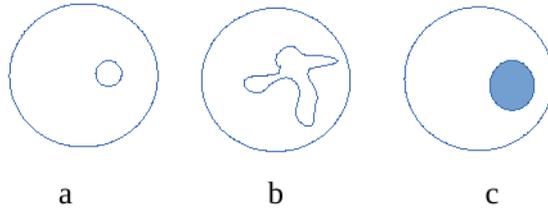


Figure 3: Time evolution of a disk of radius  $\rho_e$ .

space conservation, as time evolves from  $a \rightarrow b \rightarrow c$ , we see that within one collision time ( $t < \tau_{coll}$ ), this disk is distorted and starts stretching in both parallel and perpendicular directions. The two new length scales are:  $\rho_{long.direction} = \rho_e \cdot e^{l_{mfp}/l_c}$  and  $\rho_{short.direction} = \rho_e \cdot e^{-l_{mfp}/l_c}$  respectively. At  $t = \tau_{coll}$ , the coarse graining smears the disk to a larger scale length so to fill a bigger surface.

The distance traveled within one collision time  $\tau$  is  $l_{mfp} = v_{th} \cdot \tau$  where the collisions are treated as a discrete process taking place periodically at constant time intervals  $\tau$  with no memories between the steps. Assuming there is no memory between those time steps, the particle's motion is random and the span in the radial direction is  $\langle (\delta r)^2 \rangle \sim D_M l_{mfp}$  in one collision time  $\tau_{coll}$ . The collisionless stochastic heat diffusion coefficient in the perpendicular direction is then equal to:

$$\chi_{\perp} = \frac{\langle (\delta r)^2 \rangle}{2\tau_{coll}} \sim \frac{D_M \cdot l_{mfp}}{\tau_{coll}} = D_M \cdot v_{th} \quad (10)$$

Clearly this coefficient is independent of collisionality and collision frequency but requires coarse graining which is the essence of the irreversibility of the process.

Case 2:  $l_{ac} < l_{mfp} < l_c$ : the so called collisional regime

In this case, particles randomly undergo multiple kicks in the parallel direction within one  $l_c$ . Parallel motion is diffusive and for a time  $t \gg \tau$ , the average traveled distance  $L$  is equal to:

$$L^2 \sim \chi_{\parallel} \cdot t \sim v_{th} \cdot l_{mfp} \cdot t$$

where  $\chi_{\parallel} = v_{th} \cdot l_{mfp}$ . Radial spreading is then:

$$\langle (\delta r)^2 \rangle = 2 \cdot L \cdot D_M = 2 \cdot D_M \cdot (\chi_{\parallel} \cdot t)^{1/2}$$

In the perpendicular direction, the motion is continuous and the course graining is needed to kick the particles off the magnetic lines. The spreading caused by particles backscattering occurs at a diffusion coefficient:

$$\chi_{\perp} = \rho_e^2 \nu_e = \rho_e^2 \cdot (v_{th}/l_{mfp})$$

The backscattering modifies the perpendicular traveled distance and the radial displacement squared is now:

$$\langle (\delta r)^2 \rangle \sim D_M \cdot l_{c,\delta}$$

where  $l_{c,\delta}$  is now affected by all previous collisions the particle experienced. Noting that  $(\chi_{\parallel}/l_{c,\delta}^2 \sim 1/dt)$ , we have:

$$\frac{\langle \delta r^2 \rangle}{t} \sim \frac{\chi_{\parallel}}{l_{c,\delta}^2} \cdot D_M \cdot l_{c,\delta}$$

which gives:

$$\chi_{\perp} = D_M \cdot \frac{\chi_{\parallel}}{l_{c,\delta}} \quad (11)$$

Obviously we need to find  $l_{c,\delta}$ . To do so we need to consider the competition between two processes which balance is what ultimately sets this  $l_{c,\delta}$  length [6]. The first process is the decrease in the width experienced by the disk in the previous picture. Stochastic instability makes the width decrease exponentially (See fig.4):

$$d\delta/dl = -\delta/l_c \quad (12)$$

Here  $dl$  is the distance traveled by a particle during a time  $dt$ . The second process occurs as a natural consequence of diffusive collisions and results in an increase in the width of this structure island by an amount  $d\delta \sim (\chi_{\perp} \cdot dt)^{1/2}$  (See fig.5). But since  $1/dt = \chi_{\parallel}/(dl)^2$ , we can write:

$$d\delta \sim (\chi_{\perp}/\chi_{\parallel})^{1/2} \cdot dl \quad (13)$$

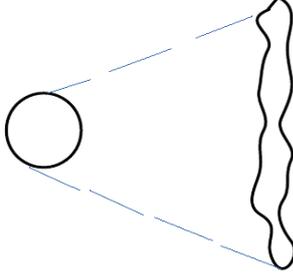


Figure 4: Width decreases as a consequence of area conservation.

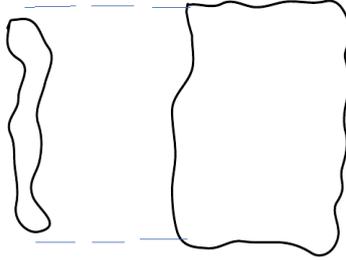


Figure 5: Width increase as a consequence of diffusion process.

Balancing Eq.(12) and Eq.(13), we get:

$$d\delta \simeq (\chi_{\perp}/\chi_{\parallel})^{1/2} dl \simeq (\delta/l_c) dl$$

The first proportionality is a direct translation of the smearing of the structure island and the second proportionality translates a thinning process. The  $\delta$  expression is now equal to:

$$\delta \sim -l_c \left( \frac{\chi_{\perp}}{\chi_{\parallel}} \right)^{1/2} \quad (14)$$

In order to convert  $\delta$  to  $l_{c,\delta}$ , we use  $\bar{k}_{\theta}$  to translate the fact that if the disk is squeezed in one direction, it should extend in another as a result of area conservation. With  $\bar{k}_{\theta}^{-1} \simeq \delta e^{l_{c,\delta}/l_c}$ , this gives:

$$l_{c,\delta} = l_c \cdot \log\left(\frac{1}{\bar{k}_{\theta} \cdot \delta}\right) = l_c \cdot \log\left(\frac{(\chi_{\parallel}/\chi_{\perp})^{1/2}}{\bar{k}_{\theta}} \cdot l_c\right) \quad (15)$$

and the  $\chi_{\perp}$  expression is:

$$\chi_{\perp} = D_M \cdot \frac{\chi_{\parallel}}{l_c \cdot \log\left(\frac{(\chi_{\parallel}/\chi_{\perp})^{1/2}}{\bar{k}_{\theta}} \cdot l_c\right)} = v_{th} \cdot D_M \cdot (l_{mfp}/l_c) \quad (16)$$

Now comparing expressions of  $\chi_{\perp}$  in the collisional and the collisionless cases, we find that the thermal diffusivity in the latter case is smaller than that

obtained for the collisionless case by a factor  $l_{mfp}/l_{c,\delta} < 1$  as one should have anticipated. Long story short, collisions act to reduce the thermal diffusivity in the perpendicular direction. These same collisions knock particles off the field lines and insure a radial transport.

## 4 An alternative route, an analytic approach

One might choose to adopt another route and try to recover the same diffusivity properties and expressions in a more systematic way by adopting a hydrodynamic perspective. Let us consider the heat flux along the perturbed magnetic field lines and split its expression into parallel and perpendicular to  $\vec{B}$  components:

$$\vec{q} = -\chi_{\parallel} \nabla_{\parallel} T \hat{b} - \chi_{\perp} \vec{\nabla}_{\perp} T$$

where  $\chi_{\parallel} \gg \chi_{\perp}$ . The unit vector  $\hat{b}$  is  $\hat{b} = \hat{z}_0 + \tilde{b}$  and represents both the unperturbed and fluctuating parts. The corresponding operator  $\nabla_{\parallel} = \partial_z + \tilde{b} \cdot \nabla_{\perp}$ .

We start with a quasi linear approximation and allow for a temperature perturbation:  $T = T_0 + \tilde{T}$ . For a stationary condition, the heat flux is divergence free and  $\nabla \cdot \vec{q} = 0$ . Expanding the expression of the heat flux  $\vec{q}$  and  $\langle q_r \rangle$  while using the corresponding expressions of  $\hat{b}$  and  $\nabla$  we have:

$$\vec{q} = -\chi_{\parallel} (\partial_z + \tilde{b} \cdot \nabla_{\perp}) (T_0 + \tilde{T}) (\hat{z} + \tilde{b})$$

and

$$\langle q_r \rangle = -\chi_{\parallel} \langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle - \chi_{\parallel} \langle \tilde{b}_r \partial_z \tilde{T} \rangle - \chi_{\parallel} \langle \tilde{b}_r \tilde{b}_r \partial_r \tilde{T} \rangle - \chi_{\perp} \partial_r \langle T \rangle \quad (17)$$

The first two terms are non linear quadratic and the third term is cubic in perturbation. If we calculate the ratio of the third to the second term, we find it equal to  $(\tilde{b}_r \cdot l_{ac}) / \Delta_{\perp}$  ( $1/\partial_z \sim l_{ac}$  and  $1/\partial_r \sim \Delta_r$ ). This ratio is ultimately equal to the  $Ku$  number. Because we are interested in low Kubo number cases, we will drop the third term cubic in the nonlinearity and consider a radial flux expression  $\langle q_r \rangle$ :

$$\langle q_r \rangle = -\chi_{\parallel} [\langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle + \langle \tilde{b}_r \partial_z \tilde{T} \rangle] - \chi_{\perp} \partial_r \langle T \rangle \quad (18)$$

which can be written as:

$$\langle q_r \rangle = -\chi_{\parallel} [\langle \tilde{b}_r \rangle \langle \tilde{b}_r \cdot \nabla T \rangle] - \chi_{\perp} \partial_r \langle T \rangle \quad (19)$$

For fluctuations to drive a nonlinear radial flux,  $\tilde{b} \cdot \nabla T$  needs to be  $\neq 0$  and the temperature needs to vary along the field lines to drive parallel heat flux. In the case of a stationary heat flux  $\nabla \cdot \vec{q} = 0$  requires the results to be

$\chi_{\perp}$  dependent. Now if we linearize the expression of  $\vec{q}$ , we get the double identity:

$$\vec{q} = -\chi_{\parallel} \partial_z^2 \tilde{T} - \chi_{\perp} \nabla_{\perp}^2 \tilde{T} = -\chi_{\parallel} \partial_z \tilde{b} \frac{\partial \langle T \rangle}{\partial r} \quad (20)$$

Working in the Fourier k-space:

$$\tilde{T}_k = \frac{ik_z \tilde{b}_k \chi_{\parallel}}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2} \cdot \frac{\partial \langle T \rangle}{\partial r}.$$

Plugging back in the nonlinear radial heat flux expression, we get:

$$\langle q_{NL} \rangle = -\chi_{\parallel} \cdot \frac{\partial \langle T \rangle}{\partial r} \cdot \sum_k \frac{\chi_{\perp} k_{\perp}^2 b_k^2}{\chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2} \quad (21)$$

with an explicit dependence on  $\chi_{\perp}$ , and:

$$\langle q_r \rangle = -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 b_k^2}{(\frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2)(1 + \frac{k_z^2}{(\chi_{\perp}/\chi_{\parallel}) k_{\perp}^2})} \quad (22)$$

The factor of the form  $1 + \alpha$  in the denominator of the previous expression is simply the auto-correlation length  $l_{ac}$ , and when the integral is calculated, we get:

$$\langle q_r \rangle = -\frac{\partial \langle T \rangle}{\partial r} \int dk_{\perp} k_{\perp}^2 \frac{(\chi_{\parallel} \chi_{\perp})^{1/2}}{k_{\perp}} \langle \tilde{b}_k^2 \rangle l_{ac} \quad (23)$$

$$\langle q_r \rangle = -\nabla \langle T \rangle \cdot k_{\perp} \cdot (\chi_{\parallel} \chi_{\perp})^{1/2} \cdot \langle \tilde{b}_k^2 \rangle \cdot l_{ac} \quad (24)$$

This translates into an effective perpendicular diffusivity:

$$\chi_{\perp} = k_{\perp} \cdot (\chi_{\parallel} \chi_{\perp})^{1/2} \cdot \langle \tilde{b}_k^2 \rangle \cdot l_{ac}$$

But  $\sqrt{\chi_{\parallel} \chi_{\perp}} \simeq \sqrt{v_{th}^2 \rho_e^2} \simeq D_B$ , so this means that the effective electron thermal diffusivity is:

$$\chi_{eff} = \frac{D_B \cdot D_M}{\Delta_{\perp}}$$

where  $\chi_{eff}$  scales with Bohm and not Spitzer diffusion and  $\Delta_{\perp} = k_{\perp}^{-1}$ . Comparing the results to those in Eq. (11) and dropping the logarithm in the expression of  $l_{c,\delta}$ , if we set  $\Delta_{\perp} = l_c \sqrt{\frac{\chi_{\perp}}{\chi_{\parallel}}}$  as a result of adopting  $\frac{\chi_{\parallel}}{l_c^2} = \frac{\chi_{\perp}}{\Delta_{\perp}^2}$ , we have

$$\chi_{eff} = \frac{\sqrt{\chi_{\parallel} \chi_{\perp}} \langle \tilde{b}_r^2 \rangle l_{ac}}{l_c (\frac{\chi_{\perp}}{\chi_{\parallel}})^{1/2}} = \frac{\chi_{\parallel} D_M}{l_c}$$

and we recover the same results obtained in [6] where  $D_M = \langle \tilde{b}_r^2 \rangle l_{ac}$ .